MATH CLUB: NUMBER THEORY PROBLEMS

NOV 5, 2017

1. A number written out as a decimal is called a *base 10 representation*, where the place values are decided by powers of 10. For example, $527 = 5 \cdot 100 + 2 \cdot 10 + 7 \cdot 1$. A similar representation can be done for powers of a different number n, resulting in a string of digits whose place values are decided by powers of n. Such a representation is called a number's *base n representation*, and is often written with a subscript, for example $51_7 = 5 \cdot 7 + 1 \cdot 1 = 36_{10}$. The digits in a base n representation will be numbers from 0 to $n-1$.

(a) Given a number k, when is the last digit of k's base 7 representation the same as the last digit of its base 10 representation?

(b) Consider a number k whose decimal (base 10) representation has six digits. What's the largest number of distinct digits k can possibly have when represented in base 7?

(c) Is there a number k whose base 10 representation has six digits but its base 7 representation's digits are all the same?

2. (a) Determine the units digit (the last, or rightmost, digit) of the base 10 representation of $1! + 2! + 3! + ... + 9!$.

(b) Determine the units digit of the base 7 representation of $1! + 2! + 3! + ... + 9!$.

3. How many pairs of positive integers are there such that the difference of their squares is 35?

4. Does there exist a rectangle with positive integer side lengths whose area and perimeter sum to 31?

5. In how many zeroes does the number 100! end?

6. Prove that, given any prime $p > 5$, there must be a number of the form 1111...11 (i.e., a string of 1s in decimal representation) that is a multiple of p. [Hint: first prove that there are two numbers of such form whose difference is a multiple of p.]

7. Without using computers or calculators, find the last 4 digits of $2^{1000} + 5^{1000}$.

8. 200 students sit in a classroom. Their teacher labels them with the numbers 1-200 (each student gets a distinct label), and lines them up in numeric order.
The teacher then writes a large number on the board in the front of the room, and tells the students that they may leave if and only if their number divides the number on the board.
Two students remain, who are sitting next to each other. What are their numbers?

9. (challenge) There are n children equally spaced around a merry-go-round with n seats, waiting to get on. The children climb onto the merry-go-round one by one (but not necessarily going in order around the circle), always using the seat in front of them and only taking a seat if it is empty. After one child climbs on and takes a seat, the merry-go-round rotates $360/n$ degrees counterclockwise so that each remaining child is again lined up with a seat. For what values of n is it possible for the children to climb on, in some order, so that everyone gets a seat? (Do remember to prove both that it's possible for the values you claim and that it's impossible for all other values.)
Source: Mathcamp 2011 Qualifying Quiz, proposed by Michael Wu, a student at Mathcamp 2009 & 2010.