School Nova
Function Equations
Alexander Kirillov, Rahul Mane

1. Prove that there is no function f from the set of non-negative integers to itself such that
$$f(f(n)) = n + 1$$
for every n.

2. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that
$$f(x) + f(t) = f(y) + f(z)$$
for all rational numbers $x < y < z < t$ that form an arithmetic progression. ($\mathbb{Q}$ is the set of all rational numbers.)

3. Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x^2 - y^2) = xf(x) - yf(y)$$
For all pairs of real numbers x and y.

4. Find all injective functions $f : \mathbb{R} \to \mathbb{R}$ satisfying
$$f(xy)(f(x) - f(y)) = (x - y)f(x)f(y)$$
(Note: injective means that distinct inputs always have distinct outputs, i.e. if $x \neq y$ then $f(x) \neq f(y)$.)

5. Does there exist a function $s : \mathbb{Q} \to \{-1, 1\}$ such that if x and y are distinct rational numbers satisfying $xy = 1$ or $x + y \in \{0, 1\}$, then $s(x)s(y) = -1$? Justify your answer. (2004 IMO Shortlist)

6. For which positive integers n does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that $f^n(x) = -x$ for all x and $f^m(x) \neq -x$ for all x and $m < n$? (Here f^n denotes f composed with itself n times; for example, $f^4(x) = f(f(f(f(x))))$.)
For which positive integers n does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that $f^n(x) = 1/x$ for all x and $f^m(x) \neq 1/x$ for all x and $m < n$?

Note: the above problems are from the following competitions: 1 is modified from 1987 IMO; 2 is from 2015 USAJMO; 3 is from 2002 USAMO; 4 is modified from 2001 IMO Shortlist; 5 is from 2004 IMO Shortlist.