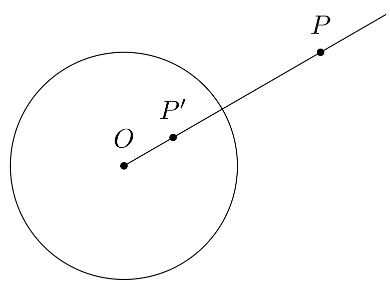
MATH CLUB: INVERSION 2

OCT 1, 2017

Inversion

Definition. Let S be a circle with center at O and radius r. *Inversion* in circle S is the transformation I of the plane that sends every point P to a point P' such that

- P' is on the ray OP
- $OP \cdot OP' = r^2$



Note that $I(O)$ is undefined.

Inversion is not a rigid motion — it doesn't preserve distances. Yet, it has a number of interesting properties summarized below.

Theorem 1. *Let I be an inversion in circle S, with center at O and radius r. Then*

1. *I send every straight line not containing O to a circle through O. Conversely, it send every circle through O to a straight line.*
2. *I sends every circle not containing O to another circle.*

Note that while I sends circles to circles, it doesn't send center of a circle to the center of transformed circle.

Theorem 2. *Let I be an inversion. Then I preserves angles: if two lines l, m intersect at point P at angle α, then $I(l), I(m)$ will intersect at point $I(P)$ at angle α, and similar if one or both lines is replaced by a circle.*

(By definition angle between two circles at intersection point P is the angle between their tangent lines at P.)

If two circles C_1, C_2 go through O and are tangent to each other at O, then $I(C_1)$, $I(C_2)$ are two parallel lines. Conversely, if l, m are two parallel lines which do not go through O, then $I(l)$, $I(m)$ are two circles that are tangent at point O.

More problems

1. Given two non-inersecting circles C_1, C_2 and a point P (outside of both circles), construct a circle through P which is perpendicular to both C_1, C_2.

2. Let us consider points of the plane as complex numebrs in the usual way: point with coordinates (a, b) corresponds to complex number $z = a + bi$. Consider the transformation of the plane given by

$$z \mapsto \frac{1}{\overline{z}}$$

where $\overline{z}$ is the complex conjugate of z: $\overline{a + bi} = a - bi$. Prove that this transformation is an inversion, with center at $(0, 0)$.

3. Find all positive integers p, q such that $p^{2017} + q$ is a multiple of pq, and p, q are relatively prime.

4. On a screen of 2017×2017 pixels, at least $2016^2 + 1$ pixels are lit. Every second, if in some 2×2 square of the screen three pixels are dark, the fourth one goes dark as well. Prove that nevertheless, at all times at least one pixels will stay lit.

***5.** For four complex numbers z_1, z_2, z_3, z_4, define their cross-ratio by

$$(z_1, z_2; z_3, z_4) = \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)}.$$

(a) Show that the cross-ratio doesn't change under the transformations $z \mapsto cz$, $z \mapsto z + c$, $z \mapsto 1/z$.
(b) Show that given three distinct complex numbers z_2, z_3, z_4, one can find a composition of these transformations which sends points z_2, z_3, z_4 to $0, 1, \infty$ respectively.
(c) Deduce that four points z_1, z_2, z_3, z_4 lie on the same line or circle if and only if the cross-ratio $(z_1, z_2; z_3, z_4)$ is real.