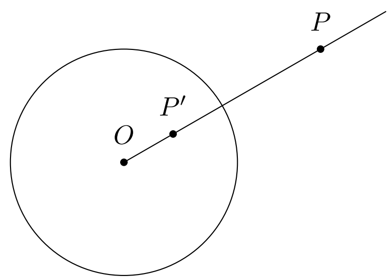
MATH CLUB: INVERSION

SEPTEMBER 24, 2017

Inversion

Definition. Let S be a circle with center at O and radius r. *Inversion* in circle S is the transformation I of the plane that sends every point P to a point P' such that

- P' is on the ray OP
- $OP \cdot OP' = r^2$



Note that $I(O)$ is undefined.

Inversion is not a rigid motion — it doesn't preserve distances. Yet, it has a number of interesting properties summarized below.

Theorem 1. *Let I be an inversion in circle S, with center at O and radius r. Then*

1. *I send every straight line not containing O to a circle through O. Conversely, it send every circle through O to a straight line.*
2. *I sends every circle not containing O to another circle.*

Note that while I sends circles to circles, it doesn't send center of a circle to the center of transformed circle.

Theorem 2. *Let I be an inversion. Then I preserves angles: if two lines l, m intersect at point P at angle α, then $I(l), I(m)$ will intersect at point $I(P)$ at angle α, and similar if one or both lines is replaced by a circle.*

(By definition angle between two circles at intersection point P is the angle between their tangent lines at P.)

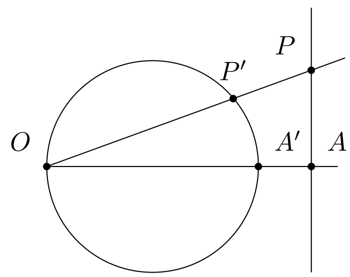
If two circles C_1, C_2 go through O and are tangent to each other at O, then $I(C_1)$, $I(C_2)$ are two parallel lines. Conversely, if l, m are two parallel lines which do not go through O, then $I(l)$, $I(m)$ are two circles that are tangent at point O.

Some problems

1. Given two non-inersecting circles C_1, C_2 and a point P (outside of both circles), construct a circle through P which is tangent to both S_1, S_2.

2. Given three circles C_1, C_2, C_3 such that C_1 and C_2 are externally tangent to each other and C_3 is outside C_1, C_2, can you construct the fourth circle C tangent to C_1, C_2, C_3? [Hint: use inversion!]

3. Prove Theorem 1.1. [Hint: use similar triangles to show that in the figure below $OA \cdot OA' = OP \cdot OP'$.]

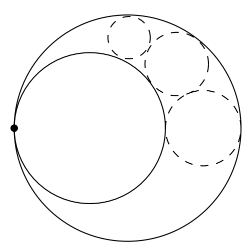


4. (a) Let C be a circle, and let O be a point outside the circle. Let l be a line through O which intersects circle C at points P, P'. Prove that then $OP \cdot OP' = r^2$, where r is the lenght of the tangent from O to circle C.

(b) In the notation of part (a), show that in this case, inversion in circle with center at O and radius r sends circle C to itself.

(c) Prove Theorem 1.2.

5. Let two circles S_1, S_2 be tangent to each other, with one circle inside the other, as shown in the figure below. Construct a sequence of circles $C_1, C_2, \ldots$, which are tangent to both S_1, S_2, and each next one is tangent to the previous (three first such circles are shown in the figure below by dashed lines). Prove that then centers of all circles C_i lie on some circle.



6. And now for something completely different... (and simple!)

You have a collection of numbers $1, 2, \ldots, 25$ written on the board. Every minute Daniil chooses a pair of numbers, erases them, and writes a new number instead: if the numbers were a, b, then he replaces them with $a + b + ab$. He repeats this until there is a single number written on the board.

What is this number?